The Lagrange points - more than simply solutions to a differential equation

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We would learn a lot more about outer space if only it were possible to position a satellite such that it were motionless in space. It could then measure changes in the magnetic field strength or the particle flow at a fixed position. But that is impossible. In order to remain truly ''still'' in one place a satellite would actually have to continue to be ''in motion''. Only in this way could it counteract all of the gravitational influences in its environment. However, there are special points in our Solar System that at least approximate the above-mentioned state: the so-called Lagrange points. At these points (almost) all of the forces involved add up to zero. In this article we want to ascertain together where the points are located and which special characteristics they possess.

Overview of references		
Astronomy	Space travel, applied astronomy (observation technology and processes)	Satellites, flight paths, mission profiles, Trojans
Physics	Gravitation, oscillations	Kinetic energy, potential energy, law of gravitation, Coriolis force, Circular motion, two-body problem, three-body problem, cosmic velocities, Lagrange points, stability of equilibrium positions, Lissajous figures
Related disciplines	Astro-mathematics, astrophysics	Equations of motion, three-body problem

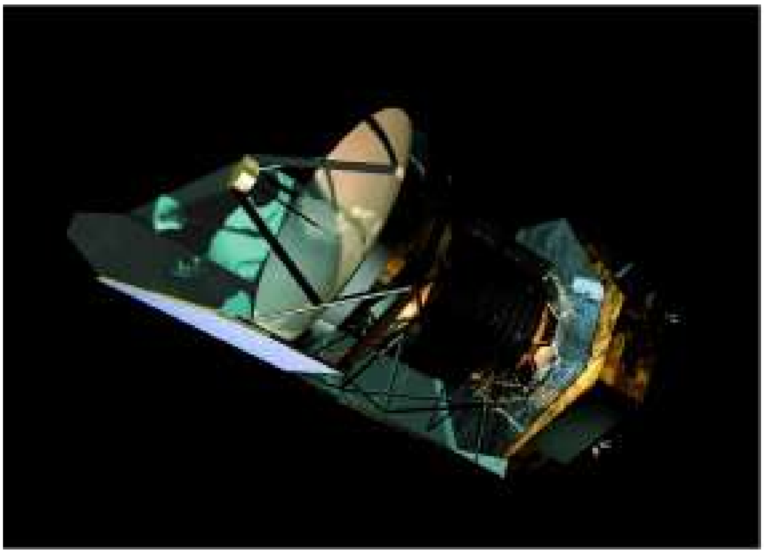

Figure 1:
The Herschel satellite (previously called 'FIRST') will be the first space observatory to cover the entire wavelength range of the far-infrared range (FIR) including the sub-millimetre range (60 - 670 micrometres). Its launch is planned for 29.2.2008. It is named after the German-British astronomer Sir Friedrich Wilhelm Herschel (1738-1822) who discovered infrared radiation in 1800. (Source: ESA, graphic: media lab)

Recommended topics for discussion: Think about exactly what it means to be 'motionless' in space. Does 'motionless' in relation to a certain point/body perhaps make more sense?
Tips: Think of various frames of reference. Is there a 'fixed' point at all in outer space?

Exercise 1: Research: What are satellites and space probes used for, and which flight paths have advantages for each?

Exercise 2: Many people have heard of the so-called *geostationary orbit* of a satellite: A satellite flies above the Earth's equator on a path that is synchronous with the Earth's rotation, meaning that it moves at the same rotational speed as the Earth. Calculate at what height such a satellite must fly above the Earth's surface.

Exercise 3: Determine the true speed of a point on the Earth's surface in connection with the geographical latitude φ of its location. Compare these values with those of other planets in our Solar System (e.g. the gas giants). What do you notice?

Law of gravitation

Newton's brilliant achievements included that he was the first person to apply the laws of physics - which were known to apply on Earth - to the movement of celestial bodies in outer space. According to a popular anecdote, the truth of which is however doubted, an apple falling from a tree led him to the idea in 1666 that terrestrial gravity could also exert an influence on the moon. He proved this supposition to be true in his famous *lunar calculation* in 1666. Twenty years later Newton introduced his theory of gravity with his magnum opus *Philosophiae naturalis principia mathematica*. He was able to verify Kepler's supposition that the attraction between two bodies decreases with the square of the distance between them, whether on Earth or in space:

$$\vec{F}_G = -G \frac{m_1 m_2}{r^2} \vec{r}_0$$

G is called the *gravitational constant* and its value is G = 6.672 $10^{-11} m^3 kg^{-1} s^{-2}$. This can be determined in an experiment, e.g. using a torsion balance as per *Cavendish* and *Eotvos*. The following applies for the value of the gravitational force:

$$F_G = -G \frac{m_1 m_2}{r^2}$$

Exercise 4: Carry out Newton's lunar calculation yourself.

Under the following link you will find a simulation of the lunar calculation: Apples are thrown parallel to the Earth's surface from a high tower at arbitrary initial speed. Play with various speeds. How high does the speed have to be for the apple to circle the Earth - like a moon? Compare the results of the simulation with your calculations (exercise 2).

Exercise 5: Extrapolate the law of gravitation assuming that a planet with mass *m* is moving on a circular path around a central body with mass M. (Tip: 3. Kepler's third law).

Cosmic velocities

By looking at the levels of cosmic velocities we can see the enormous speeds at which rockets must be launched into outer space in order to *escape the gravitational pull of the Earth or of any other planet:*

If a body with mass *m* is firstly lifted from the surface of a central body (mass *M,* radius *R)* to a certain height and secondly brought into orbit at this position, the energy required to launch it from the surface of the central body is generally provided in the form of kinetic energy $E_k = 1/2\, m\, v_0^2$.
If it is only lifted to a certain height $h = r - R$, terminal velocity $v_e = 0$, this energy is the same as the work in the central force field.

$$E_{\mathrm{kin}} = \frac{1}{2} m v_0^2 = \int_R^r -\mathrm{G} \frac{mM}{r^2} \mathrm{d}r = \mathrm{G}\, mM \left(\frac{1}{R} - \frac{1}{r}\right)$$

If the body is also required to form a circular orbit at height *r* , it must be provided with the kinetic energy for this path.

$$m \frac{v^2}{r} = \mathrm{G} \frac{mM}{r^2} \Leftrightarrow v = \sqrt{\mathrm{G} \frac{M}{r}}$$

$$\frac{1}{2} m v^2 = \frac{1}{2} m \frac{\mathrm{G} M}{r} = \mathrm{G}\, mM \frac{1}{2\mathrm{r}}$$

From this we obtain:

$$E_{\mathrm{kin}} = \frac{1}{2} m v_0^2 = \mathrm{G}\, mM \left(\frac{1}{R} - \frac{1}{2\mathrm{r}}\right)$$

First cosmic speed

This is what we call the speed at which a body moving tangentially to a planet surface (here: Earth) no longer falls back down onto the planet but instead moves at the lowest orbit around the planet *(r≈R).*

$$v_1 = \sqrt{\mathrm{G} \frac{M}{R}} = 7{,}91\ \mathrm{km/s} = 28\,400\ \mathrm{km/h}$$

If launching in an easterly direction, the Earth’s rotation adds an amount of around 0.46 km/s the orbit speed; in a westerly direction this figure must be compensated for accordingly. The orbit speed of polar paths remains uninfluenced by the Earth’s rotation.

Second cosmic speed

This is the name for the speed at which a body leaves the gravitational field of the planet (here: Earth) ($r \to \infty$).

$$v_2 = \sqrt{2 G \frac{M}{R}} = \sqrt{2}\, v_1 = 11{,}2\,\mathrm{km/s} = 40164\,\mathrm{km/h} \quad .$$

Third cosmic speed

This indicates the initial speed at which a rocket must be launched such that it has enough energy to leave the Solar System.

In order to leave the gravitational field of the Sun (beginning from the Earth) the probe requires a speed of 42.2 km/s. However the Earth already moves around the Sun at approx. 30 km/s. This means that it must be launched from the Earth at a relative velocity of at least v_R = 142.2 km/s - 30 km/s = 12.2 km/s in order to escape the pull of the Sun. In order to completely leave the Earth it requires an additional v_2= 11.2 km/s.

$$\frac{1}{2} m v_R^2 + \frac{1}{2} m v_2^2 = \frac{1}{2} m v_3^2$$

$$v_3 = \sqrt{(12{,}2\,\mathrm{km/s})^2 + (11{,}2\,\mathrm{km/s})^2} = 16{,}6\,\mathrm{km/s} = 59621\,\mathrm{km/h} \quad .$$

If a probe is launched at velocity v which is greater than the velocity actually needed, a terminal velocity v_E is left over.

$$v = \sqrt{v_2^2 + v_E^2} \quad .$$

Exercise 6: The comet probe Rosetta is supposed to reach a terminal velocity of v_E = 3.545 km/s in outer space. At what speed does Rosetta need to be launched from Earth?

We know that a somewhat greater initial speed results in a considerably greater terminal velocity.

The three-body problem

We are now sufficiently familiar with the two-body problem. We will now add an additional body, and it quickly gets complicated: The three-body problem represents one of the central issues in gravitational astronomy. As a result of the gravitational pull, forces act between the three bodies under the influence of which the bodies move around their common centre of gravity. With the aid of Newton's law of gravitation, we can calculate the forces exerted on each body by the two other bodies at a certain point in time, and the accelerations can thus be determined in terms of amount and direction. However, while there are closed solutions to the two-body problem (conic sections), this is no longer possible with the three-body problem. We can only specify general laws that apply in principle to all closed gravitational systems:

Centre-of-mass theorem - The common centre of gravity of the bodies pauses in a state of rest or it moves at constant speed.

Area principle or *principle of angular momentum* - The sum of the products of the masses and the surface speeds is constant and the total angular momentum when the bodies are moving remains constant.

Energy theorem - The sum of the body's kinetic and the potential energy remains constant.

The Lagrange points

In 1772 Joseph Louis de Lagrange (1736-1813) - one of the greatest mathematicians of the 18th century - discovered that there are five points in the area surrounding two masses circulating in space at which the gravitational forces and the centrifugal forces cancel each other out. These points are called *Lagrange points after their discoverer* (also: *libration points)*. They are labelled L1 to L5. They are the algebraic solutions of the so-called limited three-body problem, in which a large mass (e.g. the Sun) and a mass that is smaller by at least a factor of 25 (e.g. the Earth) circle around a common centre of gravity. A third body of negligible mass (e.g. a space probe) also moves around the common centre of gravity when moving in the gravitational field of the first two masses, however without exerting any influence on them. (For the mathematical derivation of the Lagrange points, see Lagrange 1 (page 16f.) and Lagrange2).

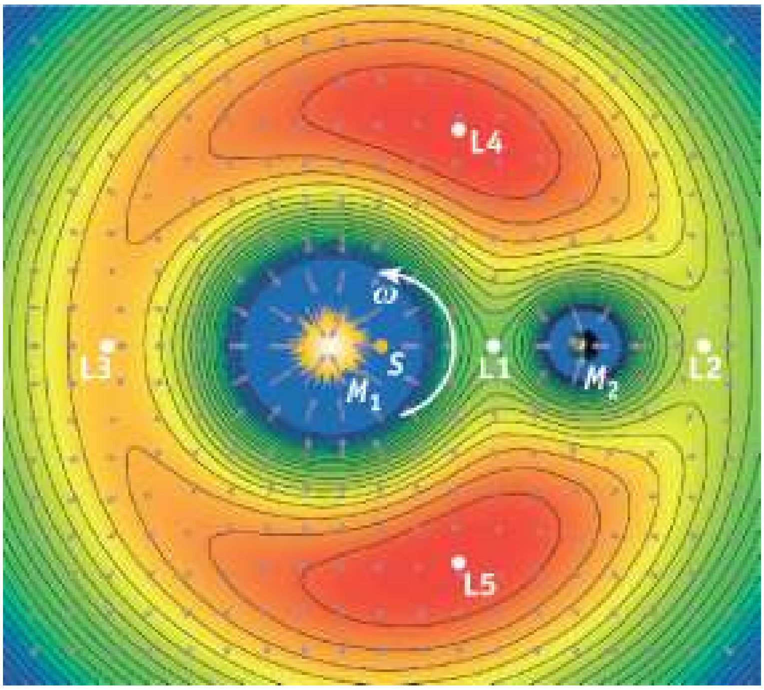

Figure 2:
The gravitational and centrifugal force field of a system *(Mi:M_2*= 5) with its five Lagrange points. Both the objects and the Lagrange points themselves move around the same centre of gravity S on an orbital plane (source: SuW 7/2003)

In our Solar System we also know of Lagrange points outside of the Sun-Earth system at which even small planets have accumulated:

In 1906 the astronomer Max Wolf from the Landessternwarte Heidelberg discovered an approximately 70 km-large asteroid in Jupiter's orbit which he called *Achilles* after a Greek hero from the Trojan war in *Homer's Iliad.* This asteroid accelerates Jupiter forward on its path around the Sun by 60°. In the same year a 147 km-large asteroid was discovered which also moves along Jupiter's path, but in this case is behind by 60°. It too was named after a hero of the Trojan War, *Patroclus*. We now know of over 1,000 asteroids that are located in the vicinity of these two points and are generally known as *Trojans*.

There is a similar case in the Saturn System: there are additional small Saturn satellites in point L4 of Saturn and Dione as well as in L4 and L5 of Saturn and Tethys.

For further information, see the Trojan simulation programme by Helmut Jahns from the VdS professional group Computer Astronomy (for details see *Trojan simulation program*).

Stability and position of the Lagrange points

In order to come to a conclusion about the stability of an equilibrium point, we need to move our space probe a little way out of its rest position in our thoughts and see what happens. Graphically this works best with the aid of Figure 3. It shows a three-dimensional illustration of the force field that a probe in the Sun-Earth system is exposed to: In the relief model the Lagrange points are each located on a horizontal position: L4 and L5 are slight ‘knolls’, L1, L2 and L3 are saddle points that are somewhat lower-lying. Moving our test object out of one of its equilibrium positions would either lead to it irrevocably ‘sliding’ into one of the wells (i.e. the probe would plummet either to the Earth or to the Sun) or the test object would leave the system. For this reason all of the equilibrium points in this static model are unstable.

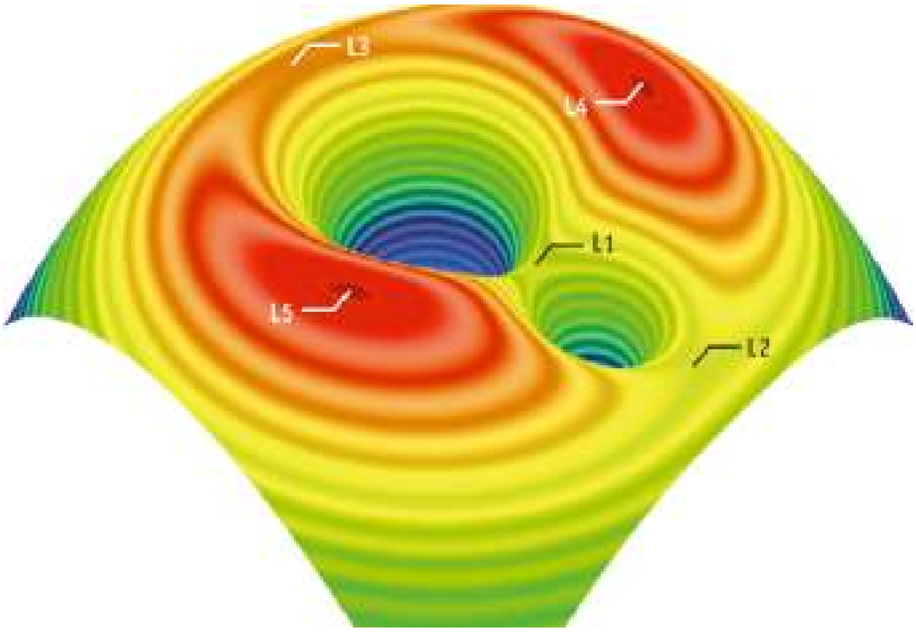

Figure 3:

Relief model of the potential characteristic for a system, comprising two bodies with a mass ratio of 5:1. (Source: SuW 7/2003)

However, because in reality we are dealing with a rotating reference system, we have the *Coriolis force* appearing in addition to the gravitational force and centrifugal force. It is not a simple function of the location, it depends on the direction and value of the speed vector of our little test object. Similar to wind flows on Earth that move in spirals and not in a straight line due to the Earth’s rotation, the Coriolis force bends the path of third body in the direction of the Lagrange point concerned. This leads to it being orbited.

L1, L2 and L3: For a body in the libration points LI, L2 or L3 the stability conditions are very strict: even if only a single additional small disturbance occurs, the paths become unstable and the small body will drift. However this happens so slowly that an almost stable position is possible on certain orbits without a large degree of manual counter-steering. Therefore the satellites concerned are left to perform a loop around the Lagrange point which only needs to be corrected every few weeks. L1 and L2 are located at a distance of around 1.5 million km from the Earth (approximately 1/100 AE), L3 is 300 million km from the Earth.

L4 and L5: Lagrange was already able to show that L4 and L5 represent dynamic equilibrium points that can be permanently circled by small masses. This is because at this point the Coriolis force makes itself felt by providing dynamic stabilisation. It sometimes occurs that bodies periodically move between L4, via L3 to L5 every once in a while. The envelope of this path is evocative of a horseshoe. Together with the two large masses, L4 and L5 each form an equilateral triangle. Because in the Sun-Earth system the system's centre of gravity almost coincides with the centre of the sun, L4 and L5 lie almost on the Earth's orbit. They each orbit the Sun with the Earth staggered at 60 degrees. They are located at a distance of 150 million km from the Earth.

Halo orbits

The flight paths of the space probes around the Lagrange points L1 and L2 are not simple Kepler paths as we know them from two-body problem; they are linked oscillations around the Lagrange point with different oscillation parameters in different directions in space. These paths are well-known from the study of oscillations and are called *Lissajous figures* after the French mathematician Jules Antoine Lissajous (1822-1880). They can be calculated prior to an expedition such that the probe moves away from the Earth on closed circular paths that do not cross each other, so-called *halo orbits.* Their dimension depends on the magnitude of the lunar orbit (.) and they can easily take six months to circulate.

Now we want to make some Lissajous figures ourselves. This can be done using a Lissajous computer programme, on an oscilloscope or mechanically. The last option is especially appealing to students. (The exact experiment set-up can be found in the file *Experiment: Lissajous figures*).

Exercise 7: Think about when it is that Lissajous figures make closed paths. Try to make non-closed figures using the pendulum experiment.

Boomerang effect

If exactly defined start conditions are perfectly adhered to after the launch phase it is possible to realise a fairly complex flight path that does not required course correction (.). In such case the probe can be left to its own devices and the flight path covers the departure and return flight as well as circulating the Lagrange point.

A manoeuvre of this sort was carried out by the space probe Genesis, which returned to Earth in September 2004 .

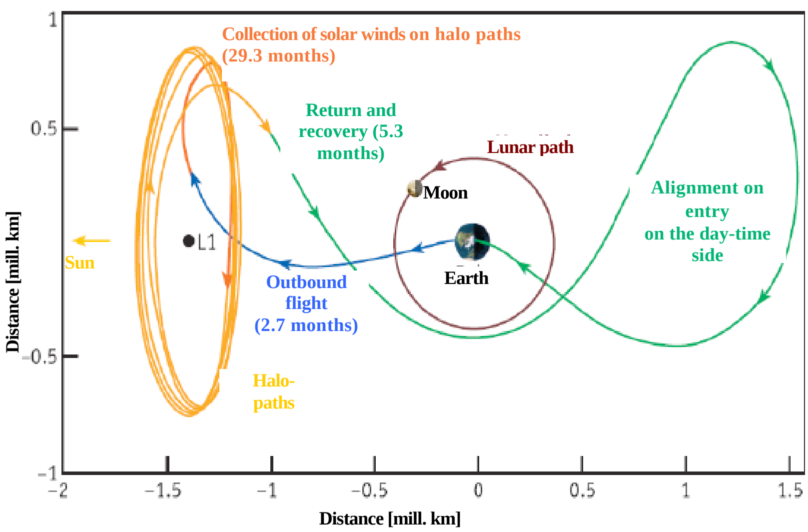

Figure 4:
Flight path of the spacecraft Genesis. (Source: SuW 7/2003)

The libration points are playing an increasingly important role in astronomy and space travel, also due to reasons simply related to cost. Therefore we will continue to hear about them often in the near future.

Exercise 8: Research on the internet to find which satellites and space probes have already been sent to the libration points of the Sun-Earth system. What scientific reasons for this can be found in the mission profiles?